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TECHNICAL NOTE

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EXPERIMENTAL INVESTIGATION OF THE NATURAL FREQUENCIES OF LIQUIDS IN TOROIDAL TANKS

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SUMMARY

Several toroidal configurations applicable to missile and space-vehicle liquid storage systems were oscillated to study the natural frequencies of the antisymmetric modes of contained liquids over a range of liquid depths and tank sizes. Natural frequencies were obtained for tank oscillations parallel to the free surface of both vertical and horizontal tank orientations.

The data are presented in terms of dimensionless parameters which are obtained by relating experimentally determined natural liquid frequencies to analytical expressions developed through consideration of the physics of the problem and from existing solutions for liquids in tanks having similar boundaries at the liquid surface. The experimental results obtained for the toroids indicate that these parameters are applicable to the prediction of the natural frequencies of fluids in toroids of general geometry and size.

INTRODUCTION

In missiles or space vehicles employing liquid-fueled propulsion systems or large volumes of liquids for life support, the responses of the system to motions of the contained liquids may greatly affect the dynamic stability of the entire vehicle. The magnitudes of these responses can increase greatly if the natural liquid frequencies are near frequencies of external periodic forces which may be induced by control impulses or by periodic structural deformations of the vehicle. It is imperative, therefore, that methods for accurately predicting the natural frequencies of these liquids in propellant tank configurations of interest be available before the tank is incorporated into any vehicle design. The natural frequencies of the antisymmetric modes are of particular interest since the liquid motions in these modes involve lateral shifts in the liquid center of gravity. Some results of experimental and analytical studies of the natural frequencies and mode shapes of liquids in spheres and right circular cylinders are given in reference 1. Recent studies, however, indicate the desirability of employing toroidal storage tanks for propellants or liquids for life-support systems in some future vehicles, but no information on the natural frequencies of liquids in such tanks is presented in the available literature.

The purpose of this paper is to report the results of an experimental investigation of the natural frequencies and mode shapes of the antisymmetric modes of liquids contained in toroidal tanks. The frequencies and mode shapes were obtained for tanks having various major and minor radii, liquid depths ranging from empty to full, and different orientations of the tanks with respect to the direction of the applied oscillation. The natural frequencies are presented as dimensionless parameters to permit application of the results to toroidal tanks of practical interest.

SYMBOLS

g	acceleration due to gravity
h	liquid depth
h_c	liquid depth in an annular right circular cylinder having the same liquid-surface geometry and yielding liquid volume equivalent to that contained in a horizontal toroid at liquid depth h (fig. 7)
J_1'	first derivative of the Bessel function of the first order and first kind
n	mode of liquid oscillation
r	minor radius of toroid (fig. 1)
R	major radius of toroid (fig. 1)
r_i, r_o	inner and outer radii of liquid surface for liquids in horizontal toroids (fig. 7)
Y_1'	first derivative of the Bessel function of the first order and second kind
ϵ_n	nth zero of J_1'

θ angle measured from the vertical to the radius of length R that terminates at the liquid surface of vertical tanks (fig. 8)

λ_n frequency parameter for nth mode of a liquid in a horizontal toroid, $\omega_n \sqrt{\frac{r_o}{g} \frac{1}{\nu_n} \frac{1}{\tanh\left(\frac{h_c}{r_o} \nu_n\right)}}$

ν_n nth root of $\frac{J_1'(\nu)}{Y_1'(\nu)} - \frac{J_1'\left(\frac{r_i}{r_o} \nu\right)}{Y_1'\left(\frac{r_i}{r_o} \nu\right)} = 0$

σ_n frequency parameter for the nth transverse mode of a liquid in a vertical toroidal tank, $\omega_n\sqrt{r/g}$

ϕ angle between the vertical and the radius to the point of intersection of the liquid surface with the peripheral circle of vertical tanks (fig. 8)

ψ_n frequency parameter for the nth longitudinal mode of a liquid in regions of a vertical toroid (fig. 1); for regions A and C,

$$\psi_1 = \omega_1 \sqrt{\frac{R + r}{g}} \sqrt{\frac{\sin \phi}{\phi}}$$

$$\psi_n = \omega_n \sqrt{\frac{R + r}{g}} \qquad (n > 1)$$

and for region B,

$$\psi_1 = \omega_1 \sqrt{\frac{R}{g}} \sqrt{\frac{\phi}{\sin \phi}}$$

$$\psi_n = \omega_n \sqrt{\frac{r}{g} \frac{1}{\epsilon_{n-1}}} \qquad (n > 1)$$

ω_n experimental natural circular frequency of oscillation of liquids in nth mode

Ω_n analytical natural circular frequency of oscillation of liquids in nth mode

APPARATUS

Description of Models

The models of the toroidal tanks studied in these tests consisted of five configurations, each of which was oriented in three different ways with respect to the direction of oscillation. These orientations are shown in the sketch in figure 1, which also includes the tabulated dimensions of the models. Orientations of the tanks for the various liquid modes were as follows:

Mode	Plane of major torus radius R	Direction of oscillation with respect to plane of major radius
Horizontal	Horizontal	Parallel
Longitudinal	Vertical	Parallel
Transverse	Vertical	Perpendicular

In all modes the direction of oscillation was in a plane parallel to the liquid surface.

All models were constructed of clear Plexiglas to permit visual observation of the liquid motion. In all cases water was used as the liquid.

Mechanical Shaker

The models were mounted on a support platform which was suspended in pendulum fashion from overhead beams. Oscillations of the models were induced by means of a mechanical shaker which was directly connected to the support platform as shown in figure 2. The mechanical shaker, described fully in reference 2, consists of a slider-crank mechanism driven by a variable-speed motor and designed so as to provide a means for conveniently varying the frequency and amplitude of the reciprocating motion applied to the platform. A tachometer, also shown in figure 2, was attached to the drive shaft of the motor and provided a means for directly obtaining the excitation frequency. The shaker had an additional design feature which permitted a rapid shutdown of the driving motion so that the liquid modes excited at a given frequency could be studied during the decay of the fluid motions.

TEST PROCEDURE

The testing technique involved inducing translatory oscillations of the models over a range of frequencies to obtain the natural frequencies of the contained liquid. The procedure was repeated over the full range of liquid depths. In measuring the lower mode natural frequencies for all models except toroid 5, the mode in question was induced by means of the mechanical shaker and, upon full development of the wave form, the platform motion was stopped and the frequencies were obtained by visually timing the low-amplitude oscillations of the liquid during the decay of the wave form. In the cases of the higher modes, for toroids 1 to 4, the excitation amplitudes were maintained at low levels and the natural frequencies were taken as those frequencies yielding maximum liquid response. In these cases, the frequencies were read directly from the tachometer. Inasmuch as toroid 5 was too large for installation on the shaker platform, the testing technique for this toroid was modified. The toroid was mounted on rollers and manually excited to induce the desired liquid mode shape. All liquid modes of this toroid were obtained by visually timing the low-amplitude liquid oscillations during the decay of the fully developed wave form. Representative mode shapes of the liquid in half-filled toroids mounted horizontally and vertically are shown in figures 3 to 5. Data were taken for all modes visually detected with sufficient clarity for their definition.

DATA REDUCTION

A sample of the test results is given in figure 6, which presents some of the experimental data taken on toroid 3. This figure shows the variation of the measured natural frequencies of the first two liquid modes in cycles per second with fullness ratio for the three tank orientations. In order to apply the results of the experimental tests to toroids of variable geometry, an extension of these results to permit the prediction of natural liquid frequencies in toroids of different dimensions is desired. The ideal method, one using dimensionless parameters which express the ratio of the experimentally determined natural liquid frequencies to simple, exact, closed-form solutions for the natural liquid frequencies, is not possible at this time since no such exact solutions are known to exist. It is possible, however, that certain alternate expressions may be derived which, when similarly applied, will yield frequency parameters independent of tank dimensions. The existence of such expressions is suggested by recognition of the fact that at various liquid depths in a toroid, the liquid has physical boundaries relatively similar to those of liquids in containers for which information concerning natural fluid frequencies is available (e.g., cylinders, spheres, and U-tubes). For some such containers, either simple, exact, closed-form solutions exist for the natural liquid frequencies, or a combination of pertinent variables has been proven experimentally to yield frequency parameters that are essentially independent of tank dimensions. The derivation of the toroidal frequency parameters based upon these expressions is presented in the following sections for each mode of oscillation. It should be noted that the frequency parameters used in the reduction of the experimental data are not the only parameters which will nondimensionalize the data. Several parameters for each mode were investigated, and those yielding the best results were selected for use.

Horizontal Modes

The frequency parameter for the horizontal modes was selected as the ratio of the experimentally determined liquid frequencies to the natural frequencies of liquids contained in an upright annular circular cylinder having dimensions r_i and r_o (the inner and outer radii of the liquid surface; see sketch in fig. 7) and a liquid depth h_c necessary to produce a liquid volume equal to the volume of liquid contained in the toroid. The exact expression for the liquid frequencies Ω_n in such an annular cylinder, given by reference 3, is

$$\Omega_n = \sqrt{\frac{g}{r_o} \nu_n \tanh\left(\frac{h_c}{r_o} \nu_n\right)} \tag{1}$$

where n is the mode of liquid oscillation, g is acceleration due to gravity and ν_n is the nth root of the equation

$$\frac{J_1'(\nu)}{Y_1'(\nu)} - \frac{J_1'\left(\frac{r_i}{r_o} \nu\right)}{Y_1'\left(\frac{r_i}{r_o} \nu\right)} = 0 \tag{2}$$

The first four roots of this equation are plotted in figure 7 as a function of $\frac{r_i}{r_o}$. The resulting parameter for the horizontal modes, denoted by λ_n, is

$$\lambda_n = \frac{\omega_n}{\Omega_n} = \omega_n \sqrt{\frac{r_o}{g} \frac{1}{\nu_n} \frac{1}{\tanh\left(\frac{h_c}{r_o} \nu_n\right)}} \tag{3}$$

Vertical Modes

The distinct differences in the geometry of the liquid boundaries for various depths in vertical toroids suggest the division of the tanks into three separate regions to permit a comparison of the experimental data for toroidal tanks of various geometry and size on the same nondimensional fullness basis. These regions are designated A, B, and C and are indicated on figure 1.

Transverse.- In vertical toroidal tanks undergoing transverse oscillations, the liquids in regions A ($0 < h < 2r$) and C ($2R < h < 2(R + r)$) have boundaries which are somewhat similar to those of liquids in spheres or horizontal cylinders undergoing transverse oscillations. It has been theoretically determined (ref. 4) and experimentally verified (ref. 1) that for a sphere or a horizontal cylinder undergoing transverse oscillations a frequency parameter σ_n of the form

$$\sigma_n = \omega_n\sqrt{\frac{r}{g}} \tag{4}$$

will insure satisfactory nondimensionalization of the natural frequencies of liquids contained in such tanks. Because of the similarity in the liquid boundary conditions, this parameter was selected for use in regions A and C.

In region B ($2r < h < 2R$) the liquid boundary is somewhat similar to that of liquids in an upright circular cylinder. The exact expression for the natural frequencies of liquids in an upright circular cylinder of radius a, given in reference 5, is

$$\Omega_n = \sqrt{\epsilon_n \frac{g}{a} \tanh\left(\frac{h}{a}\epsilon_n\right)} \tag{5}$$

where h is the depth of liquid in the cylinder and ϵ_n is the nth zero of the first derivative of the Bessel function of the first order and first kind. In the treatment of the liquid in region B as though it were contained in a cylinder of radius r, the reasonable assumption is made that the value of the equivalent liquid depth is sufficiently large to insure that $\tanh\left(\frac{h}{r}\epsilon_n\right)$ approaches unity $\left(\tanh\left(\frac{h}{r}\epsilon_1\right) \rightarrow 1.0\right.$ at $\frac{h}{r} = 1.6$; $\left.\tanh\left(\frac{h}{r}\epsilon_2\right) \rightarrow 1.0 \text{ at } \frac{h}{r} = 0.6\right)$. With this assumption, equation (5) reduces to

$$\Omega_n = \sqrt{\frac{g}{r}\epsilon_n} \tag{6}$$

It may be seen from this expression that the prime variables for nondimensionalization are g and r - the same variables that constitute the frequency parameter in regions A and C (eq. (4)). The frequency parameter for the upright transverse modes, therefore, is made to be consistent throughout the depth range; that is,

$$\sigma_n = \omega_n \sqrt{\frac{r}{g}} \tag{7}$$

Longitudinal.- The first longitudinal mode of the liquid in regions A and C is nondimensionalized by taking the ratio of the experimental natural frequency to the natural frequency of a simple pendulum whose length is equal to the distance from the center of the toroid to the center of mass of a mass distributed uniformly along an arc of the peripheral circle subtended by the liquid surface. (See figs. 8(a) and 8(c).) The expression for the natural frequency of such a pendulum is

$$\Omega_1 = \sqrt{\frac{g}{(R + r)\frac{\sin \phi}{\phi}}} \tag{8}$$

where ϕ is the angle between the vertical and the radius to the point of intersection of the fluid surface with the peripheral circle. The resulting parameter for the first mode ψ_1 is

$$\psi_1 = \frac{\omega_1}{\Omega_1} = \omega_1 \sqrt{\frac{R + r}{g}} \sqrt{\frac{\sin \phi}{\phi}} \tag{9}$$

Nondimensionalization of the higher modes in these two toroidal regions was accomplished through the consideration of the relative similarity of the liquid boundaries to those of liquids contained in spheres of radius $R + r$. The resulting frequency parameter ψ_n is then

$$\psi_n = \omega_n \sqrt{\frac{R + r}{g}} \qquad (n > 1) \tag{10}$$

First-mode liquid motion in region B is much like that in the circular-arc tube of small cross section illustrated in figure 8(b). The natural frequency of the liquid in such a tube is

$$\Omega_1 = \sqrt{\frac{g}{r}} \sqrt{\frac{\sin \theta}{\theta}} \tag{11}$$

where R is the radius to the center line of the tube and θ is the angle between the vertical and the radius R which terminates in the liquid surface. The dimensionless frequency parameter ψ_1 selected for the first mode is the ratio of the experimental natural liquid frequency to the frequency expression given by equation (11), or

$$\psi_1 = \frac{\omega_1}{\Omega_1} = \omega_1\sqrt{\frac{R}{g}}\sqrt{\frac{\theta}{\sin\theta}} \tag{12}$$

Nondimensionalization of the higher modes is accomplished, as with the transverse modes in this region, through the consideration of the relative similarity of the liquid boundaries to the boundaries of liquids contained in an upright circular cylinder. The order of the analogous cylindrical frequency is decreased by 1 since the manometer-type mode not present in the upright cylinder was considered as the fundamental mode for this region. Again, if it is assumed that the equivalent liquid depth is sufficiently large to insure that $\tanh\left(\frac{h}{r}\epsilon_n\right)$ approaches unity in the natural-frequency expression (eq. (6)), the frequency parameter is

$$\psi_n = \omega_n\sqrt{\frac{r}{g}\frac{1}{\epsilon_{n-1}}} \qquad (n > 1) \tag{13}$$

The liquid frequency parameters for the various toroidal tank orientations and the sources for the corresponding analytical expressions are summarized in table I.

DATA PRESENTATION AND DISCUSSION OF RESULTS

The frequency parameters are plotted as a function of fullness ratio $h/2r$ for the horizontal toroidal tanks and as a function of region fullness ratio $\left(\text{region A, } h/2r\text{; region B, } \frac{h - 2r}{2(R - r)}\text{; region C, } \frac{h - 2R}{2r}\right)$ for the vertical toroidal tanks. Data for each mode of a given configuration are presented in this form in figures 9 to 11. The data obtained for the higher modes of the small models were limited because the mode shapes could not be clearly defined in some cases.

Modes of Horizontal Toroids

The experimental data for the first four modes of liquids in five horizontal toroidal tanks are presented in figure 9 in terms of the frequency parameter λ_n. The values of this frequency parameter are plotted as a function of the fullness ratio. The figure indicates that at a given fullness ratio, the frequency parameter for a given mode is the same for all toroids examined. It appears, therefore, that the frequency parameter λ_n is independent of the tank dimensions (both actual size and geometric ratio R/r) and applicable for predicting the natural liquid frequencies in any horizontal toroidal tank.

It is of interest that, except at the near-full and near-empty conditions, the values of λ_n are in the neighborhood of unity, which indicates that the analytical expression for the frequency (eq. (1)) is a good approximation to the natural frequency of the liquid in horizontal toroids. Both this expression and the frequency parameter derived therefrom may be readily utilized to obtain the liquid frequencies by using the values of ν_n presented in figure 7 for the first four modes.

Transverse Modes of Vertical Toroids

The results obtained from the transverse oscillations of the vertical toroids are presented in figure 10. The frequency parameter σ_n is shown as a function of fullness ratio of the three toroidal regions for the first two liquid modes. As in the case of the liquid modes in horizontal toroids, the nondimensional data obtained for a given transverse mode of the vertical toroids may be represented by one curve. It appears, therefore, that the resulting curve is applicable for the prediction of the natural liquid frequencies in vertical toroids of any size and geometry undergoing transverse oscillations.

Longitudinal Modes of Vertical Toroids

Liquid frequency-depth relationships for the longitudinal modes of vertical toroids are presented in figure 11 in terms of the frequency parameter ψ_n. The values of this parameter for each of the three regions are plotted separately as a function of the region fullness ratio. The results for region A are presented in figure 11(a) and indicate that for a given fullness ratio, the frequency parameter for the first mode ψ_1 is the same for all toroids examined. The figure also shows that the values of ψ_1 are near unity throughout the region, which indicates that the expression given by equation (8) essentially predicts the first longitudinal liquid frequency in region A of vertical tanks.

The frequency parameter developed for the higher liquid modes in region A, based upon an expression involving the variables necessary to nondimensionalize satisfactorily the transverse modes of spheres and horizontal cylinders, is also presented as a function of region fullness ratio in figure 11(a). The data show that this parameter, ψ_n (where $n > 1$), while dimensionless, is dependent on the ratio R/r, but independent of tank size. The effect of R/r on the second mode, however, appears to be negligible. The faired data of the higher modes show that the value of the frequency parameter at a given liquid depth increases with this ratio R/r.

The results of the frequency data obtained in region B are shown in figure 11(b) and indicate that the frequencies for the three modes examined in this region are readily nondimensionalized by the parameters selected. The natural-frequency parameter ψ_n $(n = 1, 2, 3)$ has the same value for all toroids examined at a given region fullness ratio and a given mode. Thus it may be concluded that the parameters are independent of the tank dimensions and applicable in the prediction of the longitudinal liquid frequencies in the midregion of vertical toroids of practical interest. Furthermore, since the parameters are near unity except at the near-full condition, the analytical expressions for the frequencies may be used to obtain approximate natural frequencies throughout most of the depth range of region B.

The variation of the frequency parameters ψ_n with fullness ratio for region C is presented in figure 11(c). As was the case for region A, the frequency parameters selected appear to be sufficient to reduce the data to a family of curves indicating the dependency of the natural frequencies on R/r. Thus far, no effective parameter has been found which is useful for combining the effects of variations of both tank size and geometry in this region. The data for a given geometric ratio may be extended directly to geometrically similar tanks.

CONCLUSIONS

Frequency data have been obtained and parameters synthesized for liquids in toroidal tanks of different sizes and geometric ratios R/r (where R is the major radius and r the minor radius of the toroid). These parameters are developed for tanks oriented in three ways with respect to the direction of oscillation and have been found to be independent of tank size and in most cases independent of the geometric ratio R/r. These parameters are believed to be applicable in the prediction of the natural frequencies of fluids in toroids of practical interest in missile and space-vehicle liquid-storage systems.

The natural frequencies of liquids in horizontal toroids in general may be accurately predicted by the analytical expression for the modal frequencies of liquids in annular cylinders having the radial dimensions of the toroidal liquid surface and a depth yielding a volume equivalent to the volume of liquid in the toroid.

The transverse modes of liquids in the upper and lower regions of vertical toroids behave in a manner similar to modes of liquids in spheres, whereas the modes in the midregions behave in a manner analogous to modes of liquids in deep upright circular cylinders.

For the longitudinal modes of liquid in vertical toroids no method has been found to achieve complete nondimensionalization with respect to both size and geometry, but as expected, it was found in all cases that natural-frequency data for geometrically similar toroids can be rendered dimensionless and thus the effects of tank size can be isolated. The data presented indicate that the degree of dependence of the natural frequencies on R/r is a function of both liquid depth and the order of the liquid mode.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., July 15, 1960.

REFERENCES

1. McCarty, John Locke, and Stephens, David G.: Investigation of the Natural Frequencies of Fluids in Spherical and Cylindrical Tanks. NASA TN D-252, 1960.

2. Silveira, Milton A., Maglieri, Domenic J., and Brooks, George W.: Results of an Experimental Investigation of Small Viscous Dampers. NACA TN 4257, 1958.

3. Okhotsimskii, D. E.: Theory of the Motion of a Body With Cavities Partly Filled With a Liquid. NASA TT F-33, 1960.

4. Budiansky, Bernard: Sloshing of Liquids in Circular Canals and Spherical Tanks. Jour. Aero/Space Sci., vol. 27, no. 3, Mar. 1960, pp. 161-173.

5. Lamb, Horace: Hydrodynamics. Sixth ed., Dover Publications, 1945, p. 440.

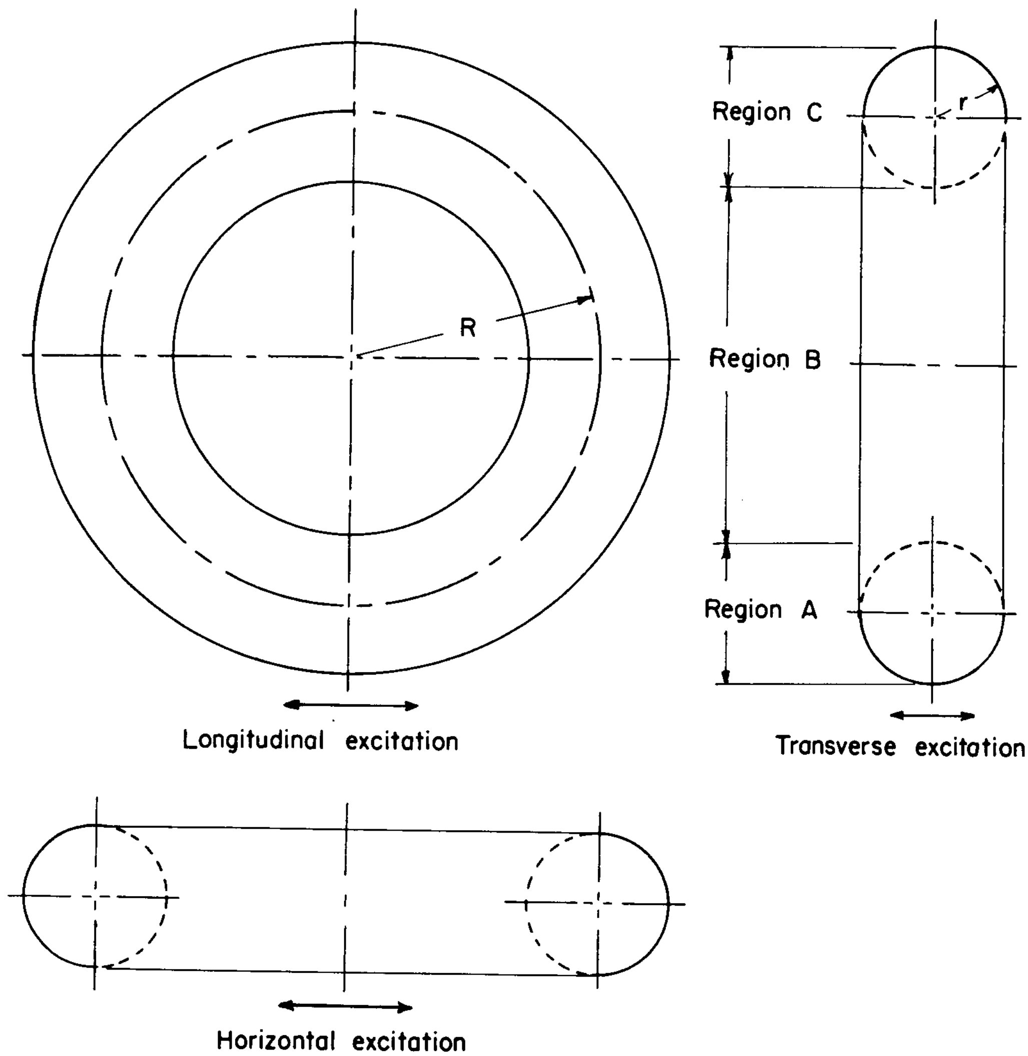
TABLE I.- SUMMARY OF NONDIMENSIONAL FREQUENCY PARAMETERS

FOR LIQUIDS IN TOROIDAL TANKS

Orientation	Region	Mode	Parameter	Source
Horizontal	All	All	$\lambda_n = \omega_n \sqrt{\frac{r_o}{g} \frac{1}{\nu_n} \frac{1}{\tanh\left(\frac{h_c}{r_o} \nu_n\right)}}$	Annular circular cylinder
Vertical transverse	A, C	All	$\sigma_n = \omega_n \sqrt{\frac{r}{g}}$	Sphere of radius r
	B	All	$\sigma_n = \omega_n \sqrt{\frac{r}{g}}$	Circular cylinder of radius r
Vertical longitudinal	A, C	First	$\psi_1 = \omega_1 \sqrt{\frac{R + r}{g}} \sqrt{\frac{\sin \phi}{\phi}}$	Simple pendulum
		All $n > 1$	$\psi_n = \omega_n \sqrt{\frac{R + r}{g}}$	Sphere of radius $R + r$
	B	First	$\psi_1 = \omega_1 \sqrt{\frac{R}{g}} \sqrt{\frac{\theta}{\sin \theta}}$	Circular-arc tube
		All $n > 1$	[a] $\psi_n = \omega_n \sqrt{\frac{r}{g} \frac{1}{\epsilon_{n-1}}}$	Circular cylinder of radius r

[a]For convenience, values of ϵ_n are listed below:

$\epsilon_1 = 1.841$
$\epsilon_2 = 5.331$
$\epsilon_3 = 8.536$
$\epsilon_4 = 11.706$



Toroid	R, in.	r, in.
1	4.0	4.0
2	7.9	3.9
3	11.9	5.9
4	11.9	3.9
5	16.0	8.0

Figure 1.- Sketch showing orientation of toroidal tank and dimensions of test configurations.

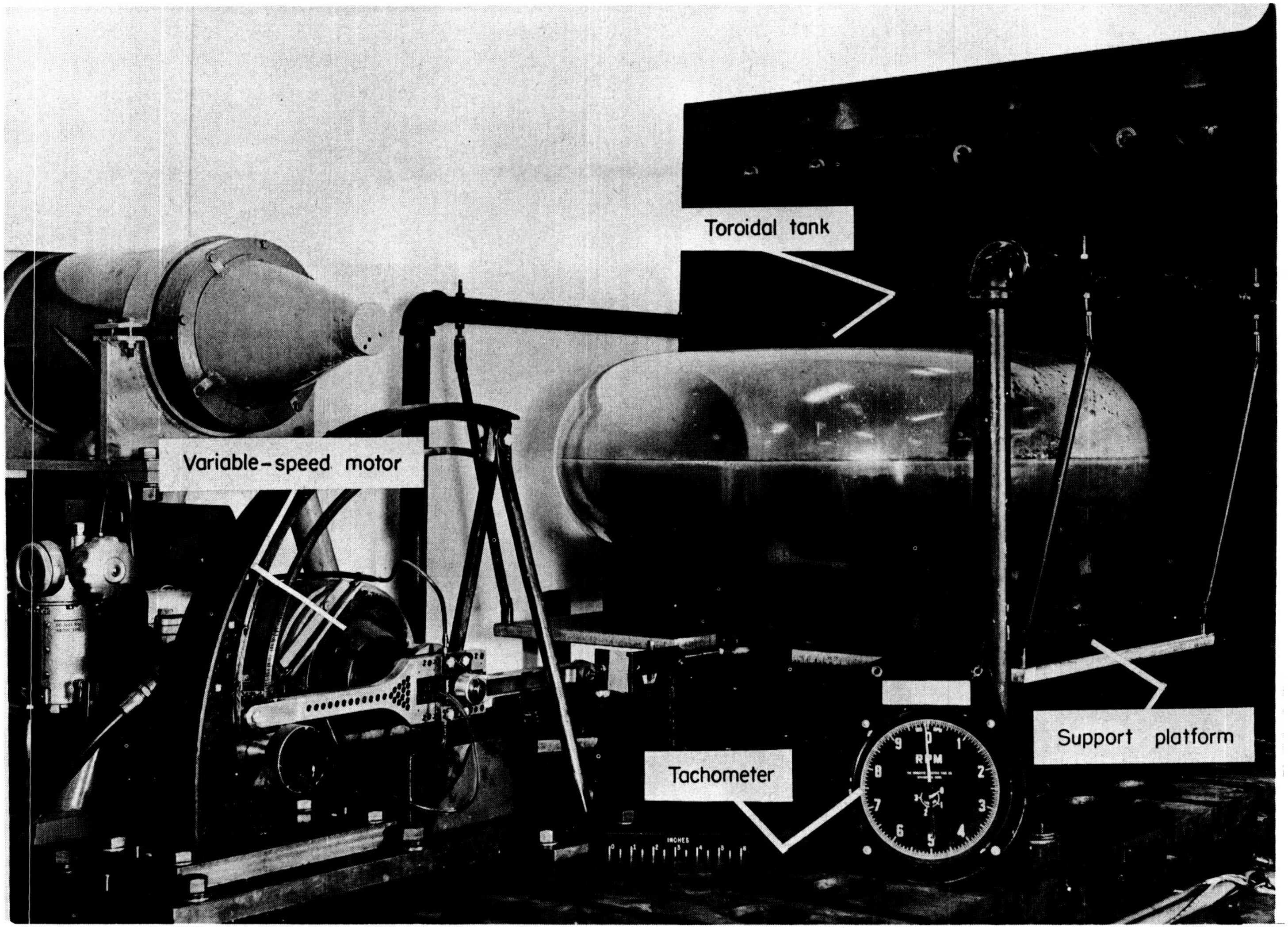


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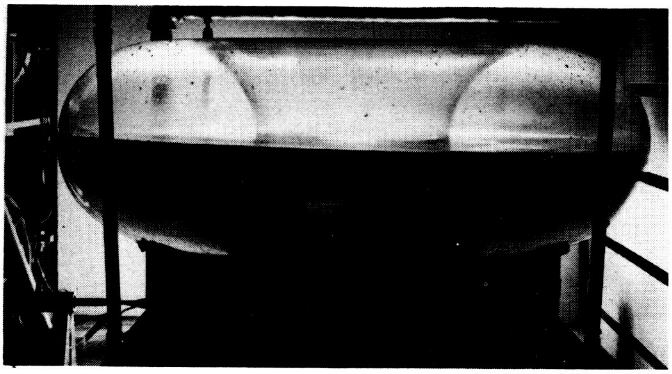
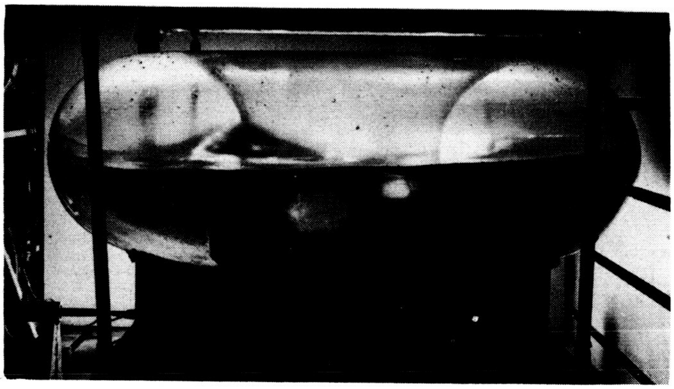
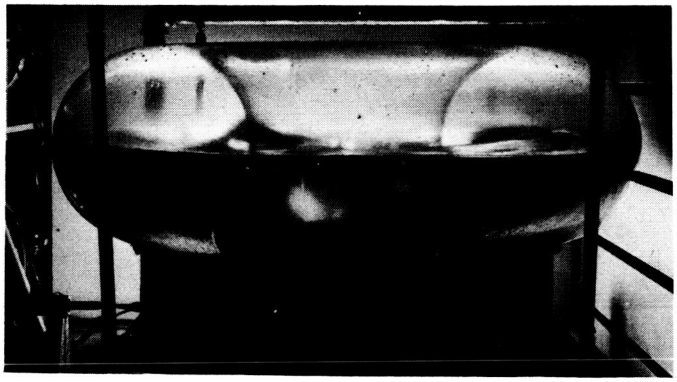
Figure 2.- Test apparatus for mechanically exciting toroidal tanks.



(a) First mode.

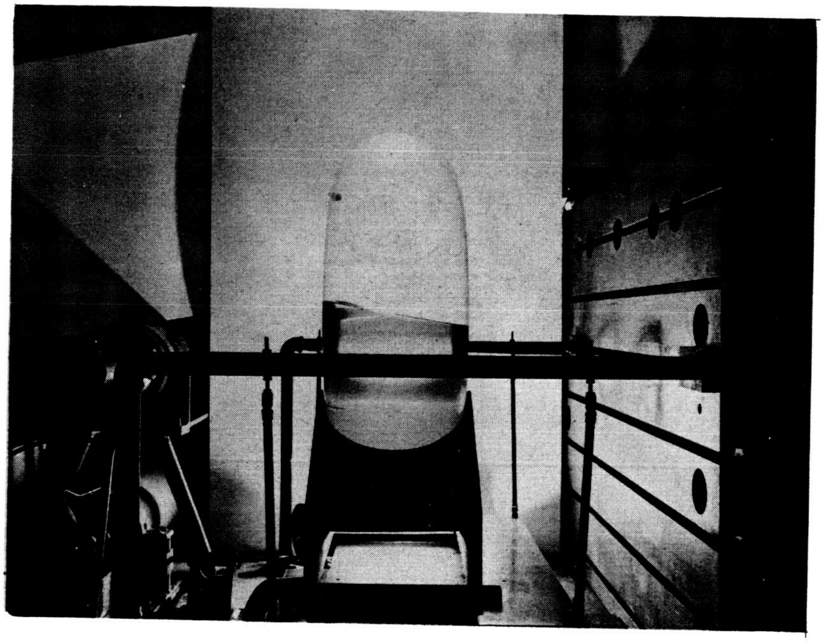


(b) Second mode.

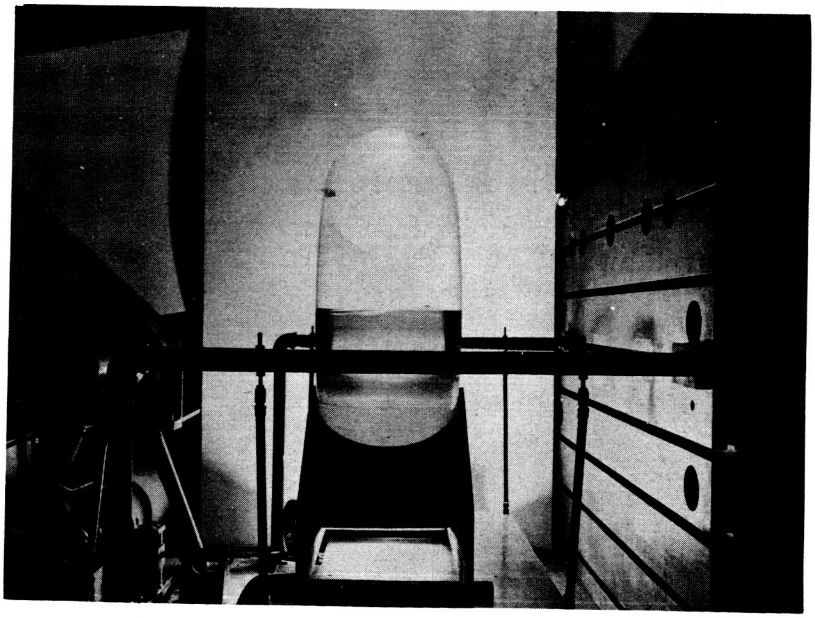


(c) Third mode. L-60-4306

Figure 3.- First three modes of liquid in a horizontal toroid.

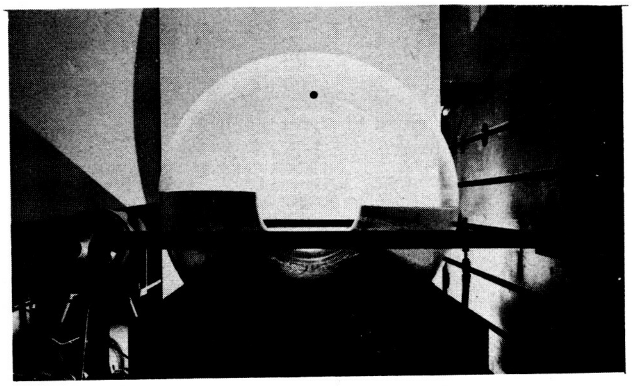


(a) First mode.

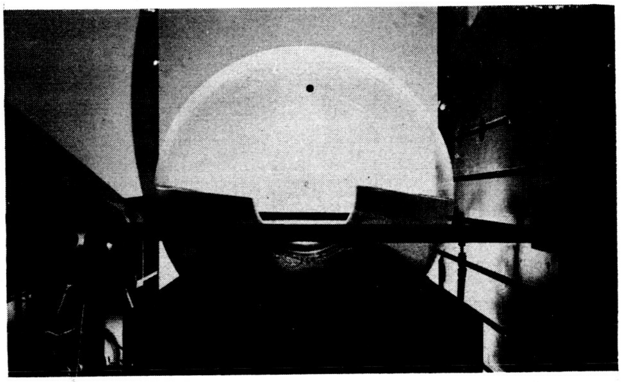


(b) Second mode. L-60-4307

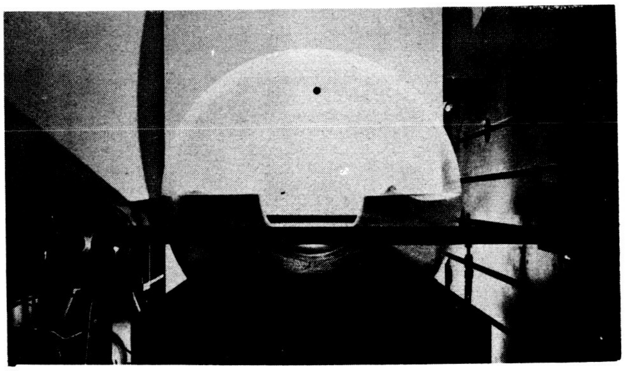
Figure 4.- First two transverse modes of liquid in a vertical toroid.



(a) First mode.

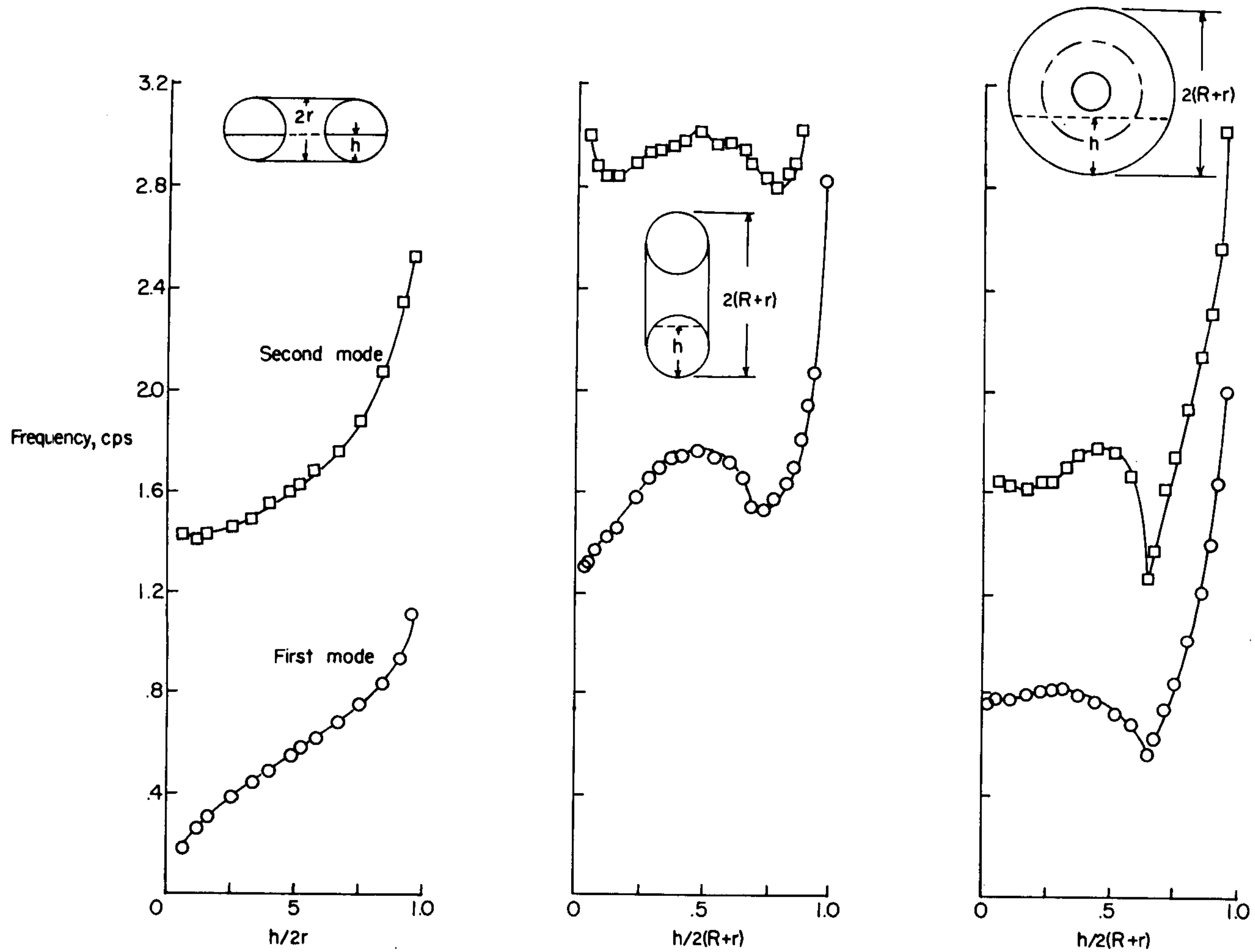


(b) Second mode.



(c) Third mode. L-60-4308

Figure 5.- First three longitudinal modes of liquid in a vertical toroid.



(a) Horizontal. (b) Transverse. (c) Longitudinal.

Figure 6.- Variation of liquid natural frequency with fullness ratio for three orientations of a toroidal tank. (Toroid number 3.)

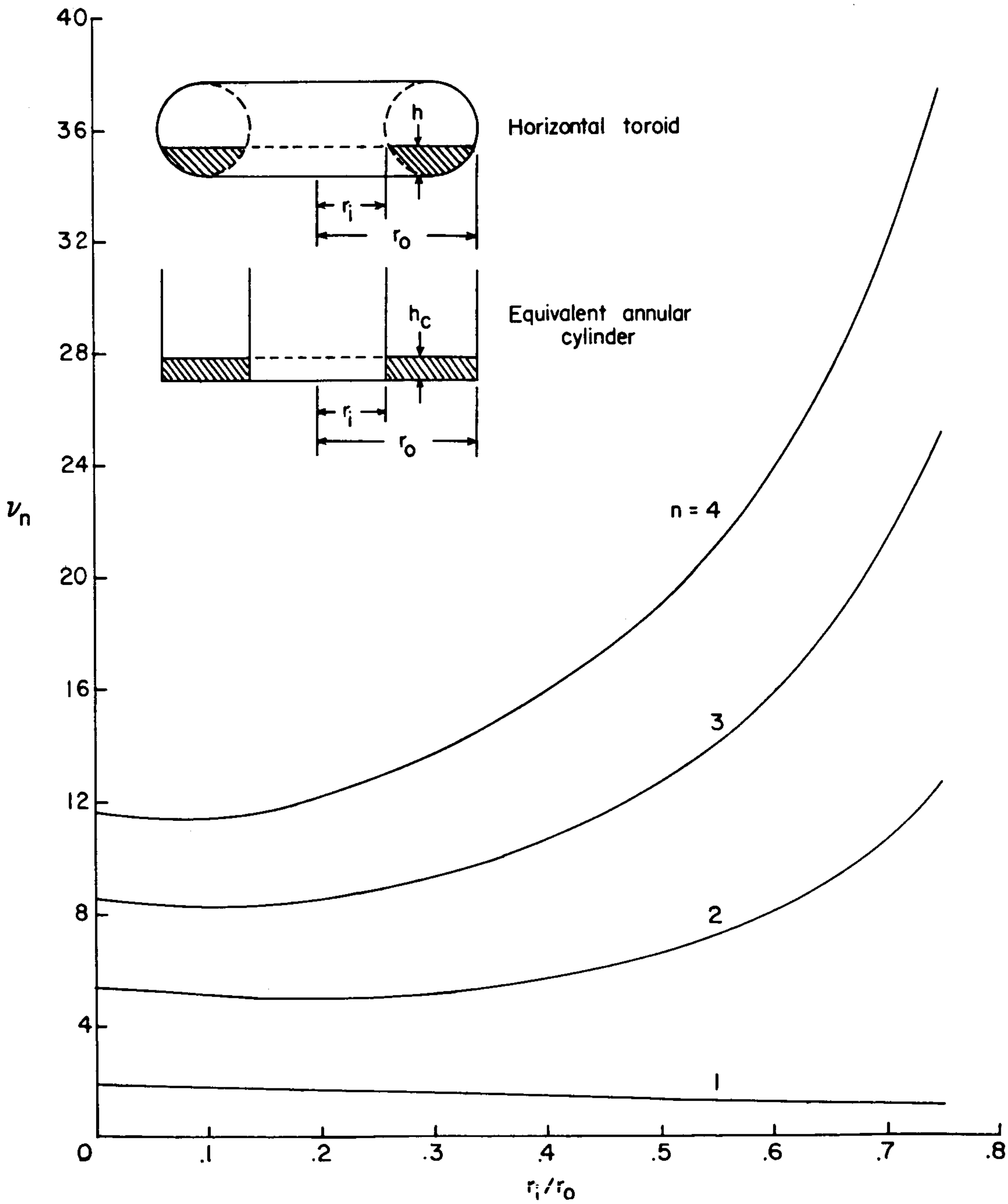
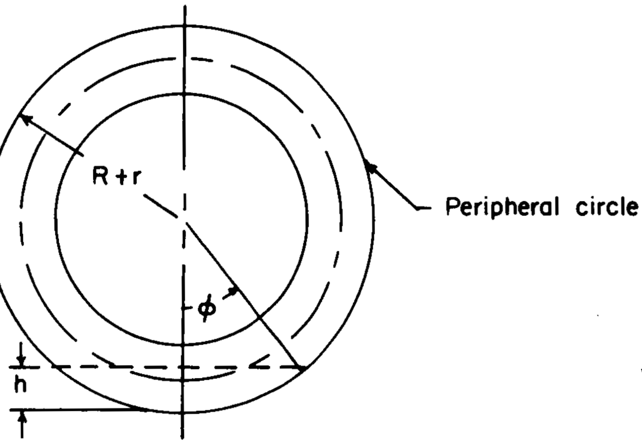
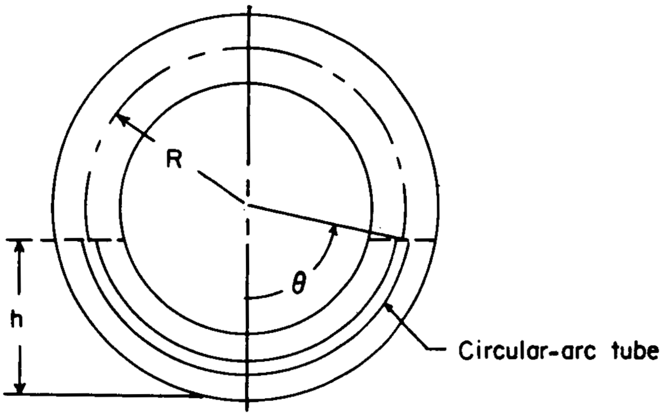


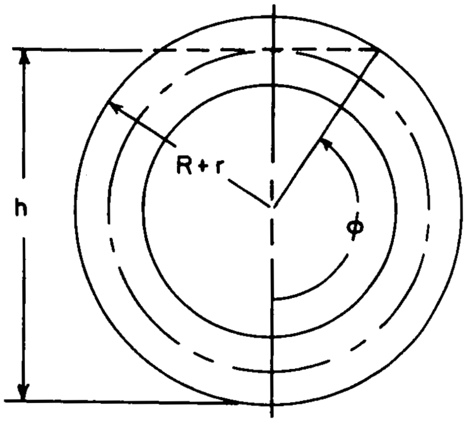
Figure 7.- First four roots of the equation $\dfrac{J_1'(\nu)}{Y_1'(\nu)} - \dfrac{J_1'\left(\dfrac{r_i}{r_o}\,\nu\right)}{Y_1'\left(\dfrac{r_i}{r_o}\,\nu\right)} = 0.$



(a) Region A.



(b) Region B.



(c) Region C.

Figure 8.- Sketch defining symbols used in treatment of longitudinal liquid modes in vertical toroidal tanks.

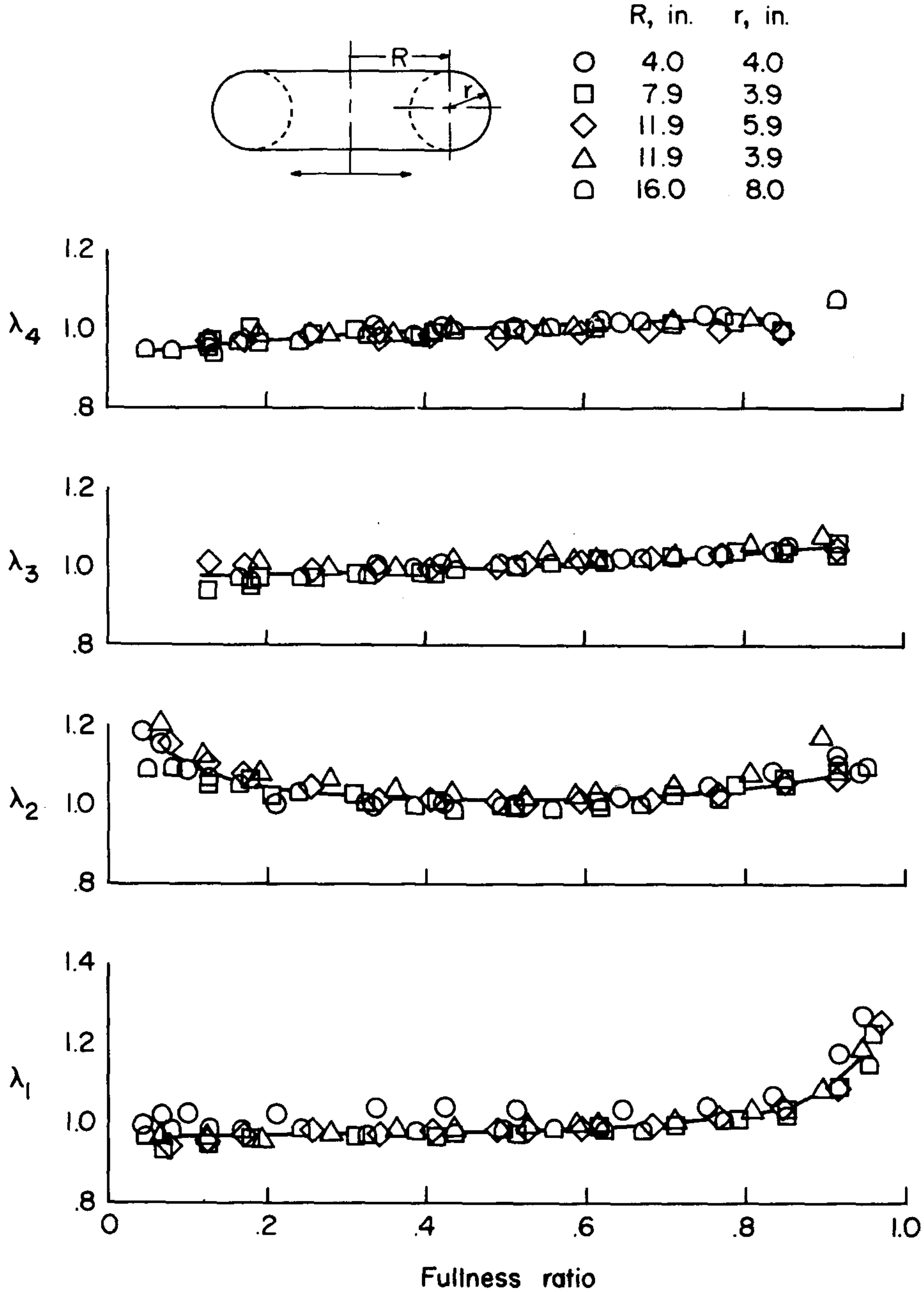


Figure 9.- Variation of liquid frequency parameter $\lambda_n = \omega_n \sqrt{\frac{r_o}{g} \frac{1}{\nu_n} \frac{1}{\tanh\left(\frac{h_c}{r_o} \nu_n\right)}}$ with depth for horizontal toroidal tanks.

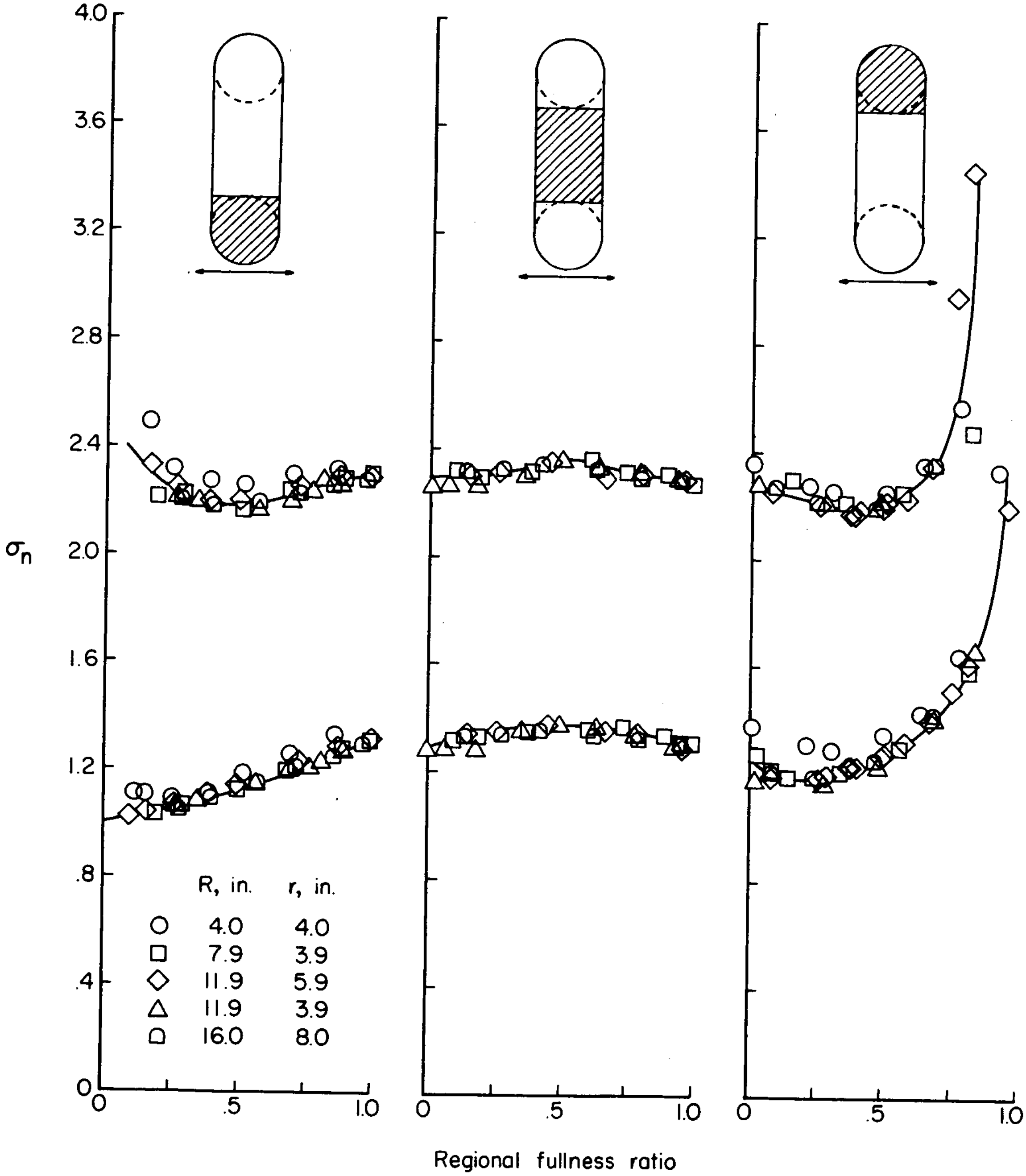
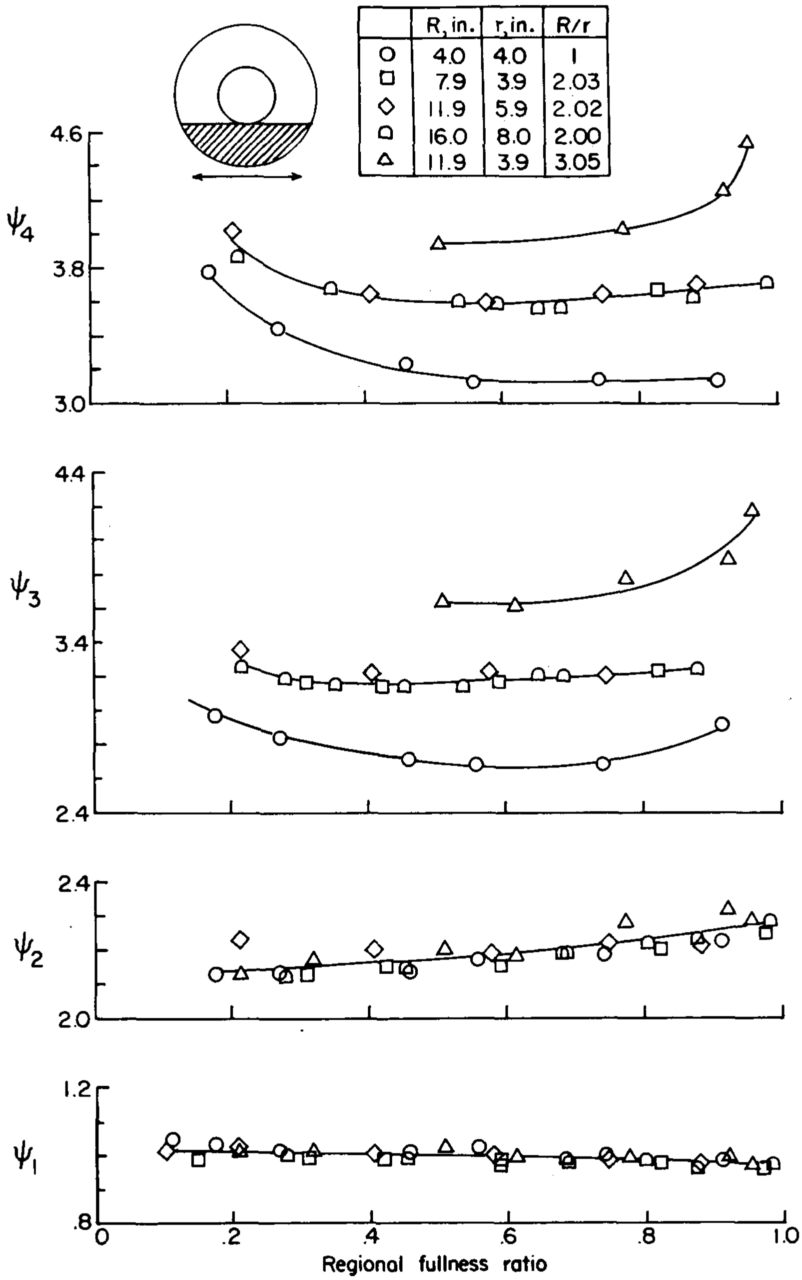
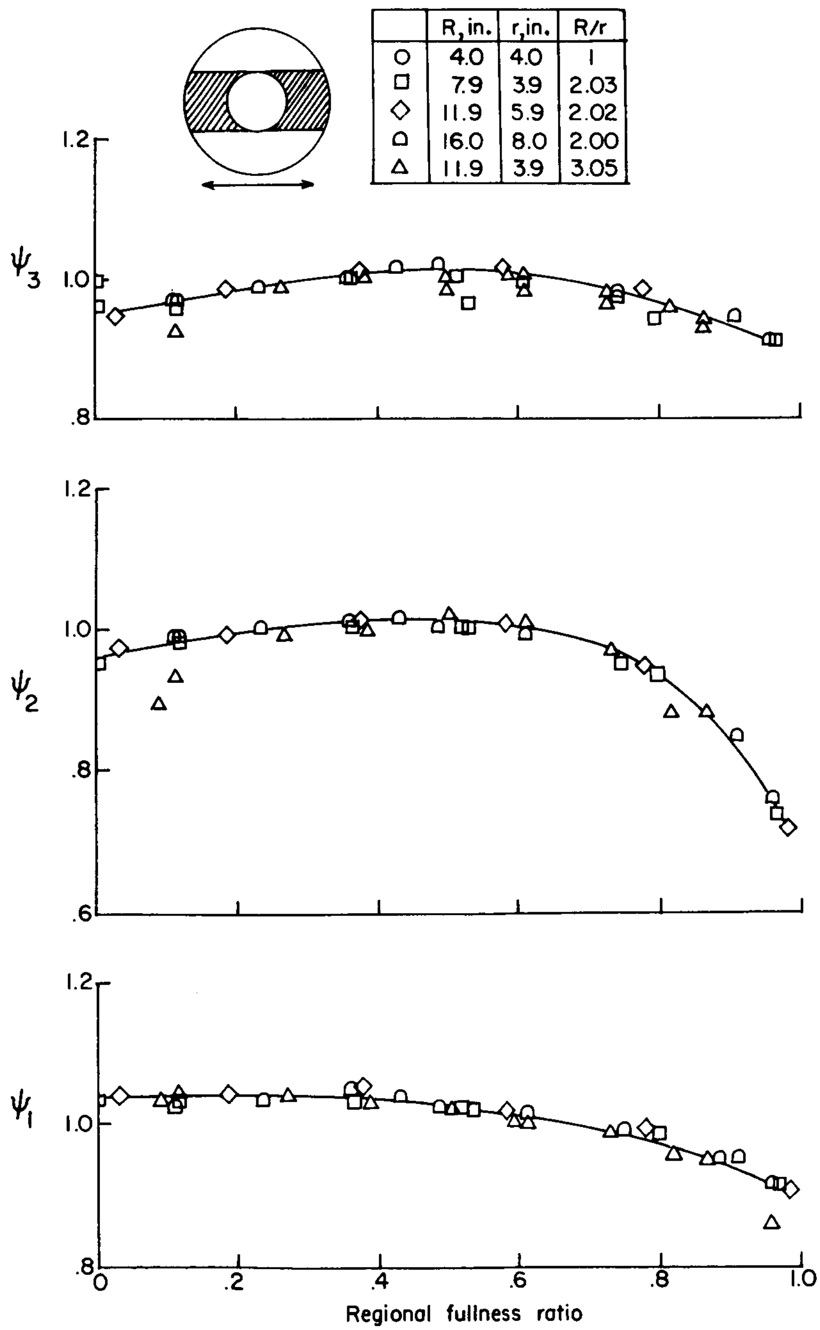


Figure 10.- Variation of liquid frequency parameter $\sigma_n = \omega_n \sqrt{\frac{r}{g}}$ with depth for transverse modes of vertical toroidal tanks.



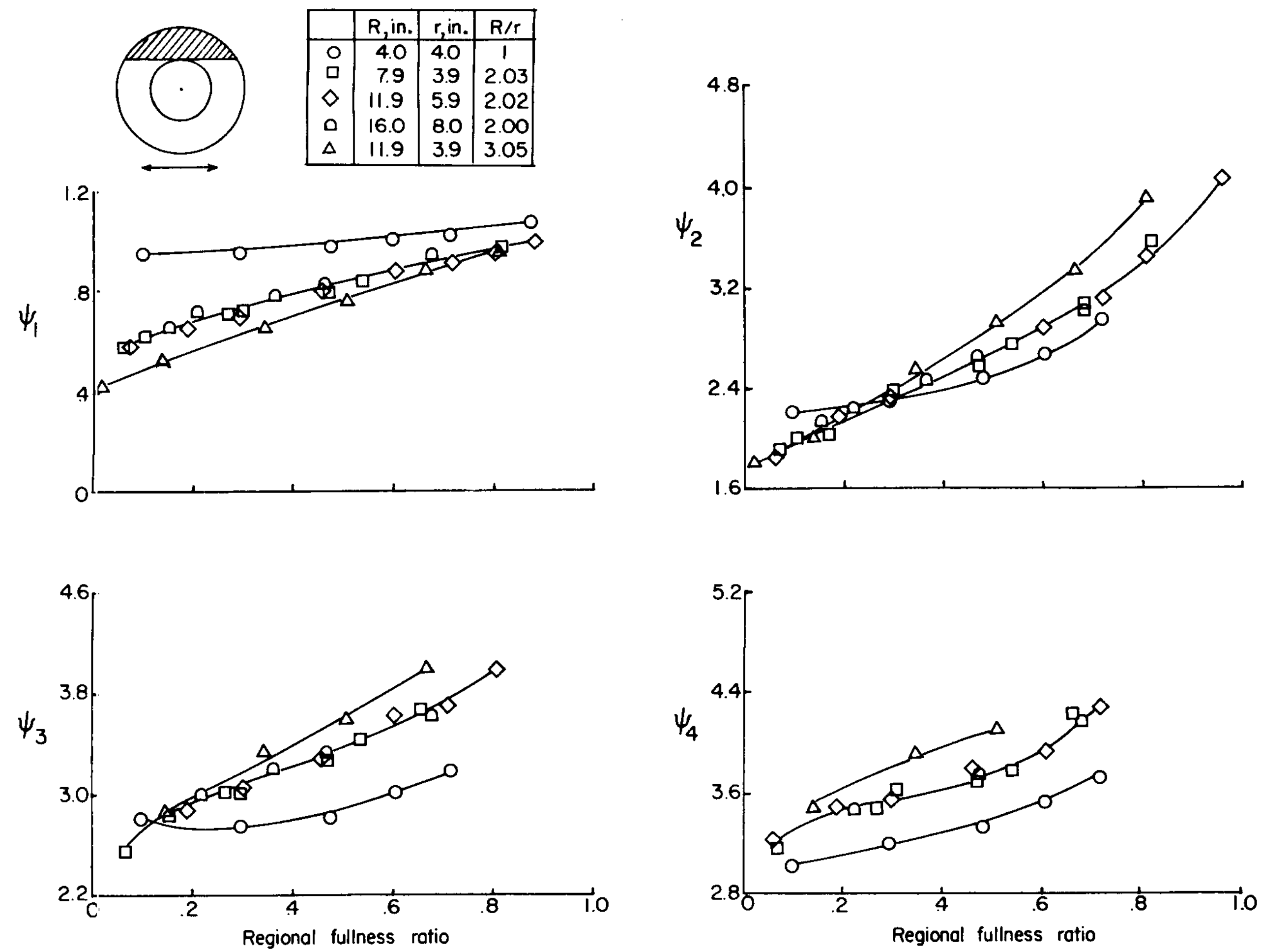
(a) Region A. $\psi_1 = \omega_1\sqrt{\frac{R + r}{g}}\sqrt{\frac{\sin \phi}{\phi}}$; $\psi_n = \omega_n\sqrt{\frac{R + r}{g}}$ when $n > 1$.

Figure 11.- Variation of liquid frequency parameters with depth for longitudinal modes of vertical toroidal tank regions.



(b) Region B. $\psi_1 = \omega_1 \sqrt{\frac{R}{g}} \sqrt{\frac{\phi}{\sin \phi}}$; $\psi_n = \omega_n \sqrt{\frac{r}{g} \frac{1}{\epsilon_{n-1}}}$ when $n > 1$.

Figure 11.- Continued.



(c) Region C. $\psi_1 = \omega_1 \sqrt{\frac{R + r}{g}} \sqrt{\frac{\sin \phi}{\phi}}$; $\psi_n = \omega_n \sqrt{\frac{R + r}{g}}$ when $n > 1$.

Figure 11.- Concluded.